Barcelona GSE
EPP-COMP Brush-up courses 2020-21
Problem Set 2

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Below you will find the collection of exercises taken from *Essential Mathematics for Economic Analysis (Sydsæter, Hammond and Strøm, 2016).* Most of exercises provided are below the level expected from a high school graduate and represent the basis that we will built upon in the following days.

The problem set will be due to 10 September and should be uploaded to the corporate platform by 11:30.

For handing in the problem set, **YOU CAN:**

- **SCAN** your solutions and send them to me
- Print or Insert images in your solutions

YOU CANNOT:

- have an illegible handwriting if you are to give a hard copy
- send photos of you solutions

PROBLEMS:

Problem 1. Write down the system of equations when $n = 4$, $m = 3$, and $a_{ij} = i + 2j + (-1)^i$ for $i = 1, 2, 3$, $j = 1, 2, 3, 4$, while $b_i = 2^i$ for $i = 1, 2, 3$

Problem 2. A firm produces non-negative output quantities $z_1, z_2, ..., z_n$ of

n different goods, using as inputs the non-negative quantities $x_1, x_2, ..., x_n$ of the same n goods. For each good i ($i = 1, ..., n$), define $y_i = z_i - x_i$ as the net output of good i, and let p_i be the price of good i. Let $\mathbf{p} = (p_1, p_2, ..., p_n)$(the **price vector**), $\mathbf{x} = (x_1, x_2, ..., x_n)$ (the **input vector**),$\mathbf{y} = (y_1, y_2, ..., y_n)$(the **net output vector**) and $\mathbf{z} = (z_1, z_2, ..., z_n)$ (the **output vector**).

- Calculate the firm's revenue and its costs.
- Show that the firm's profit is given by the inner product $\mathbf{p} \cdot \mathbf{y}$. What if $\mathbf{p} \cdot \mathbf{y}$ is negative?

Problem 3. For what values of u and v does

$$\begin{pmatrix} (1-u)^2 & v^2 & 3 \\ v & 2u & 5 \\ 6 & u & -1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & u \\ v & -3v & u-v \\ 6 & v+5 & -1 \end{pmatrix}?$$

Problem 4. Consider the three matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & 5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Find a matrix $\mathbf{C}$ satisfying $(\mathbf{A} - 2\mathbf{I})\mathbf{C} = \mathbf{I}$
- Is there a matrix $\mathbf{D}$ satisfying $(\mathbf{B} - 2\mathbf{I})\mathbf{D} = \mathbf{I}$?

Problem 5. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of order n, prove that, in general

- $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) \neq \mathbf{A}^2 - \mathbf{B}^2$
- $(\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B}) \neq \mathbf{A}^2 - 2\mathbf{AB} + \mathbf{B}^2$

Find a necessary and sufficient condition for equality to hold in each case (this will be convinient upon solving OLS).

Problem 6. Prove the last rule of transpostion seen in the lectures, namely $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$. Then if $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are matrices for which the given products are defined, show that

$$(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3)' = \mathbf{A}_3'\mathbf{A}_2'\mathbf{A}_1'$$

And generalize to product of n matrices. (*Hint: write out the matrices and check that they coincide for each c_{ij}, in the second and third parts use the rule for 2 matrices and the associative law to generalise it*)

Problem 7. Use Gaussian elimination to discuss what are the possible solutions of the following system for different values of a and b:

$$\begin{aligned} x + y - z &= 1 \\ x - y + 2z &= 2 \\ x + 2y + az &= b \end{aligned}$$

Problem 8. Calculate the following determinants

(a) $\begin{vmatrix} 3 & 0 \\ 2 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} a & a \\ b & b \end{vmatrix}$ (c) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$ (d) $\begin{vmatrix} 3^t & 2^t \\ 3^{t-1} & 2^{t-1} \end{vmatrix}$

Problem 9. Find the solutions to the equation $\begin{vmatrix} 2-x & 1 \\ 8 & -x \end{vmatrix} = 0$

Problem 10. given the matrix

$$\begin{pmatrix} a & 1 & 0 \\ 0 & -1 & a \\ -b & 0 & b \end{pmatrix},$$

find numbers a and b such that $\mathrm{tr}(\mathbf{A}) = 0$ and $|\mathbf{A}| = 12$ ($\mathrm{tr}(\mathbf{A})$ is the sum of the diagonal elements.)

Problem 11. let

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

calculate $\mathbf{X}'\mathbf{X}$ and $|\mathbf{X}'\mathbf{X}|$.

Problem 12. let

$$\mathbf{A}_a = \begin{pmatrix} a & 2 & 2 \\ 2 & a^2+1 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

calculate $|\mathbf{A}_a|$ and $|\mathbf{A}_1^6|$.

Problem 13. Solve the following systems of equations by using the inverse matrix

(a) $\begin{array}{rcrcr} 2x & - & 3y & = & 3 \\ 3x & - & 4y & = & 5 \end{array}$

(b) $\begin{array}{rcrcr} 2x & - & 3y & = & 3 \\ 3x & - & 4y & = & 5 \end{array}$

Problem 14.

(a) if $\mathbf{A}$,$\mathbf{P}$ and $\mathbf{D}$ are square matrices such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, show that $\mathbf{A}^2 = \mathbf{P}\mathbf{D}^2\mathbf{P}^{-1}$.

(b) Show by induction that $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$.

Problem 15. Suppose that $\mathbf{X}$ is an $m \times n$ matrix and that $|\mathbf{X}'\mathbf{X}| \neq 0$. Show that the matrix

$$\mathbf{A} = \mathbf{I}_m - \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'$$

is idempotent - that is, $\mathbf{A}^2 = \mathbf{A}$.